SU(2) chiral Yang–Mills model on a lattice. Witten anomaly eaten by the lattice

Corneliu Sochichiu
Bogoliubov Laboratory
of Theoretical Physics
JINR
141980 Dubna
Moscow Region
Russia
e-mail: sochichi@thsun1.jinr.dubna.su

Abstract

Using a regularisation of the chiral SU(2) Yang–Mills model by an infinite number of Pauli–Villars fields the doubler free gauge invariant lattice model is constructed. Its continuum limit provides the recently proposed mechanism for cancelation of Witten anomaly.

Formulation a model containing chiral fermions on a lattice is a long standing problem due to the phenomenon of fermion doubling [1]. It was proposed by Wilson to cure the doubling by addition of an extra term to the naive lattice action which produces large masses of the order of cut off to doubling states. This eliminate doublers in continuum limit but it destroys the chiral invariance as well. It also is the source of chiral anomaly in the continuum limit. In some cases, however, such as QCD in the limit of zero mass u and d quarks, SM or other nonanomalous models this destruction is too "strong". In this case one still needs additional "fine tuning" of the model in order to approach the chiral limit. The counter-terms needed for the "fine tuning" are not known *a priori*, and require a careful (non-perturbative) computation (for a recent review see e.g. [2]). From the other hand, no local

lattice formulation preserving the non-anomalous chiral symmetry for finite lattice spacings is known in present.

Moreover, there exists a theorem due to Nielsen and Ninomiya [3] which states that under certain circumstances such as locality, hypercubic and gauge invariance and hermicity there are no chiral fermions on a lattice.

From the other hand among the nonanomalous models there exist ones given by the real representations of gauge groups. For such models one can formulate a lattice description satisfying all conditions of Nielsen–Ninomiya theorem but with no doubling in continuum limit. This is possible because both chiralities in real representations are equivalent. For example for a model describing a chiral fermion in n-dimensional representation of $SO(n)$ one can write down the gauge invariant Wilson term as follows

$$\frac{1}{2}a\psi^{\mathrm{T}}C\Delta\psi + \text{h.c.},$$

where a is the lattice spacing, C is the Dirac charge conjugation matrix, Δ is the lattice Laplace operator and "T" stands for transposed spinor.

In case of $SU(2)$ model described by the Lagrangian

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4g^2}F_{\mu\nu} + \overline{\psi}_+\widehat{\nabla}\psi_+ \\ P_-\psi_+ &= 0,\ P_\pm = \frac{1}{2}(1\pm\gamma_5) \end{aligned} \qquad (1)$$

where ψ_+ belongs to the fundamental representation of $SU(2)$ one can construct a field

$$\overline{\psi}^c_+ = \psi_+^{\mathrm{T}}C\cdot i\sigma_2, \qquad (2)$$

where C is the same Dirac charge conjugation matrix and σ_2 is Pauli σ_2-matrix (here it plays the role of $SU(2)$ charge conjugation matrix). One can see that $\overline{\psi}^c_+$ defined in such a manner has the same chirality but transforms by conjugate action of the gauge group as initial field ψ_+. One can try to write down the gauge invariant Wilsonian term in analogy with $SO(n)$ case but since the product $C\cdot i\sigma_2$ is symmetric this term will be identically zero due to anticommuting properties of the fermionic field. As we will see later it is connected with Witten anomaly. If the number of fermionic fields were even there would be no problem since one can take the crossing product of different fields solving the problem. But one can re-define the fields in this case to show that in fact one deals with half number of Dirac fermions.

In what follows we will adopt a different strategy. In continuum one can regularize this model by introducing an infinite number of Pauli–Villars (PV) fields [4] as follows

$$\mathcal{L}_{\rm reg} = \overline{\psi}_+ \widehat{\nabla} \psi_+ + \sum_{r=1}^{\infty} \overline{\psi}_r (\widehat{\nabla} + Mr) \psi_r \tag{3}$$

where ψ_r are Dirac spinors with Grassmannian parities $(-)^r$. Regularizations of this type were used to prove the gauge invariance of the continuum limit of lattice models with Wilson fermions [5], Smit–Swift model [6] and for chiral gauge invariant lattice formulation of the $SO(10)$ unified model with SLAC fermions [7][1]. The same regularization was used recently for the construction of the representation for global Witten anomaly in continuous $SU(2)$ chiral YM model [9].

Let us return to the lattice model. We will introduce the following regularized lattice action

$$\begin{aligned} \mathcal{L}_{reg} \;=\;& \overline{\psi}_+ \left(\widehat{D} + \frac{1}{2\Lambda^2} \{ \widehat{D}, \Delta \} \right) \psi_+ + \\ & \sum_{r=1}^{\infty} \left\{ \overline{\psi}_r \left(\widehat{D} + \frac{1}{2\Lambda^2} \{ \widehat{D}, \Delta \} \right) \psi_r + Mr \overline{\psi}_r \psi_r \right\} \end{aligned} \tag{4}$$

where $\widehat{D}$ is the naive lattice Dirac operator given by

$$\left(\widehat{D} \psi \right)(x) = \frac{1}{a} \sum_{\mu} \gamma_\mu \left(U_\mu(x) \psi(x + a e_\mu) - \psi(x) \right) \tag{5}$$

here a is the lattice spacing and e_μ is the unite vector in the x_μ direction, $U_\mu(x)$ is the lattice gauge field corresponding to the link connecting x and $x + ae_\mu$, Δ in the eq.(4) is the lattice Laplace operator the same (up to an extra factor a^{-1}) as used in the Wilson term and M and Λ are new cut off parameters. The action (4) generates the following Feynman rules

$$s_r \;=\; \frac{\sum_\mu \gamma_\mu P_\mu(p) + Mr}{P^2(p) + M^2 r^2}, \quad r \neq 0 \tag{6}$$

$$s_0 \;=\; P_+ \frac{\sum_\mu \gamma_\mu P_\mu(p)}{P^2(p)}, \tag{7}$$

[1] An alternative way may consist in considering second scale regularization using a coarser lattice, for a recent review see [8], and references therein.

for propagator and

$$V^{n}_{\mu_1 \ldots \mu_n} \sim \frac{\partial^n}{\partial k_{\mu 1} \ldots \partial k_{\mu n}} \sum_{\alpha} \gamma_\alpha P_\alpha(k)|_{k=\frac{1}{2}(p+q)}, \tag{8}$$

for the vertices. In the above equations $P_\alpha(p)$ is given by

$$P_\alpha(p) = \frac{1}{a} \sin p_\mu a \left(1 + \frac{1}{\Lambda^2 a^2} \sum_\alpha (1 - \cos p_\alpha a) \right). \tag{9}$$

If $n = 1$ in the vertex we call it a simple one and if $n > 1$ we call it multiple one.

Let us show that in the framework of the perturbation theory one will have the contribution of doubling states vanishing in the continuum limit if the cutoffs M and Λ go to infinity as $\frac{1}{\sqrt{a}}$.

To analyze the continuum limit let us show as an example calculation of a diagram with n simple external legs. Generalization to multiple legs is straightforward. The Feynman integral corresponding to the diagram looks as follows

$$\begin{aligned} I^{i_1 \ldots i_n}_{\mu_1 \ldots \mu_n}(k_1, ..., k_n) \quad = \quad & \tau^{i_1 \ldots i_n} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 p}{(2\pi)^4} \mathrm{tr} P_+ V_{\mu_1}(p, p + k_1) s(p + k_1) ... \\ & ... V_{\mu_n}(p - k_n, p) s(p) + \\ & \sum_{r=1}^{\infty} (-)^r \mathrm{tr} V_{\mu_1}(p, p + k_1) s_r(p + k_1) .... V_{\mu_n}(p - k_n, p) s_r(p) \end{aligned} \tag{10}$$

where $\tau^{i_1 \ldots i_n}$ is the group factor.

Function $P_\alpha(p)$ given by the eq.(9) near the edge of the Brillouin zone behaves as follows

$$P_\alpha(p') \approx \frac{2\xi}{\Lambda^2 a^2} p'_\alpha + O(1), \tag{11}$$

where $2\xi = \sum_\mu (1 + (-)^{n_\mu})$, and quantities n_μ and p' are defined by the following

$$-\frac{\pi}{2a} < p'_\mu \equiv p_\mu - \frac{\pi}{a} n_\mu \leq \frac{\pi}{2a}$$

At the center of the Brillouin zone $P_\alpha(p)$ has the usual expansion

$$P_\alpha(p) \approx p_\alpha + ...$$

where dots stay for terms vanishing as a^2 and faster.

The propagator and the vertex at the edge of the Brillouin zone behaves like this

$$s_r(p') \approx \frac{1}{\frac{2\xi}{\Lambda^2 a^2}\hat{p'} - Mr}, \qquad \hat{p'} \equiv {p'}_\mu \gamma_\mu \tag{12}$$

$$V_\mu \approx \frac{2\xi}{\Lambda^2 a^2}\gamma_\mu. \tag{13}$$

One can see that if there were no PV regularization one would have near the edge of the Brillouin zone almost the same integrand (the factors $\frac{2\xi}{\Lambda^2 a^2}$ cancel out) except the mass term which here vanishes as $\frac{\Lambda^2 a^2}{2\xi} M \to 0$.

Since one have the PV regularization the situation changes dramatically. Indeed, the leading part of the integrand behaves like (hereafter we omit primes for p'),

$$\begin{aligned} &\sim \sum_r (-)^r \frac{1}{(P^2(p) + M^2 r^2)^n} \left(P'(p)\right)^n P^n(p) \sim \qquad (14)\\ &\sim \left(\frac{\partial}{\partial P^2(p)}\right)^{n-1} \frac{\pi}{MP^2(p)\sinh(\pi P(p)/M)} \left(P'(p)\right)^n P^n(p) \end{aligned}$$

The regularization produces exponential cut off for zones where $P(p) \gg M \sim \frac{1}{\sqrt{a}}$. As lattice spacing goes to zero and $\Lambda \sim M \sim \frac{1}{\sqrt{a}}$ to infinity one will have the cut off "starting" from values of $p \sim \frac{1}{\sqrt{a}}$ and "finishing" by the values of p approaching the Brillouin zone edge as $|p - \frac{\pi}{a}| \sim \sqrt{a}$. Indeed from eqs (11,14) one can see that for lattice spacings a small enough the asymptotic behavior of the integrand near the Brillouin zone edge is as follows

$$\sim \left(\frac{\Lambda^2 a^2}{2\xi}\frac{\partial}{\partial p^2}\right)^{n-1} \frac{\pi}{M\left(\frac{2\xi}{\Lambda^2 a^2}\right)^2 p^2 \sinh\left(2\xi\pi^2 p/M\Lambda^2 a^2\right)}, \tag{15}$$

while in the center of the zone integrand behaves like

$$\sim \left(\frac{\partial}{\partial p^2}\right)^{n-1} \frac{\pi}{Mp^2 \sinh\left(\pi^2 p/M\right)}. \tag{16}$$

From the above equations one can see that the integral over central zone tends to the correct continuum value (after extraction of UV divergencies) while integration over the strip close to the edge of the Brillouin zone decrease

with a going to zero as $\sim a^2$. In the domain of momenta between these two extrema i.e. when p is departed from the center by more then value of M and from the edge by $M\Lambda^2 a^2 \sim \sqrt{a}$ the integrand decays exponentially and its contribution to the integral vanishes in the continuum limit.

In fact to be rigorous in the analysis of the behavior of the Feynman integral near the edge of the Brillouin zone one cannot neglect the external momenta k since the loop momentum p here becomes small. But as can be shown including external momenta in the analysis cannot affect the conclusion.

As a result for any diagram in perturbation theory one have the contribution of the doubling states vanishing in the continuum limit.

From the other hand it is known that $SU(2)$ chiral YM model suffers from a global anomaly [10]. As it was shown by Witten one cannot globally define the sign of the fermionic determinant once topologically nontrivial gauge transformations which can change the sign of the determinant are considered.

In our case, however, it seems everything to be all right. One has gauge invariant regularized action with doubler's contribution suppressed in the continuum limit. And no gauge noninvariant effect like change of the determinant sign can be produced here. The problem is that the above constructed lattice regularization is equivalent to the model considered by Witten only perturbatively.

To illustrate this consider eigenvalue problem for (modified) chiral Dirac operator $\widehat{\mathcal{D}} = \left(\widehat{D} + \frac{1}{2\Lambda^2}\{\widehat{D}, \Delta\}\right)$. Charge conjugation relates both chiralities in such a way that one can introduce scalar product such that chiral Dirac operator is symmetric. Let $\{\lambda_i\}$ be the set of its eigenvalues. Then one has for the following expression for the determinant (see ref.[9])

$$\det \widehat{\mathcal{D}} = \prod_i \frac{|\lambda_i|}{\lambda_i} \tanh\left(\frac{\pi\,|\lambda_i|}{2M}\right) \tag{17}$$

one can see from this that the large eigenvalues ($|\lambda_i| \gg M$) are cut off but not their signs. The same happens with the contribution of the doubling states. Arguments from the perturbation theory shall tell us that $\frac{|\lambda_i|}{2M} \to \infty$ where λ_i are eigenvalues corresponding to unwanted (doubling) states. But when they are cut off they leave their signs alive in eq. (17). From the other hand from general reasons (e.g. Nielson–Ninomiya theorem) we know that

total number of the minus signs in eq. (17) is an even one[2]. This would prevent the determinant from being negative.

Indeed, for a sufficiently small lattice spacing one can write down the regularized determinant as follows (see Appendix)

$$\det \widehat{\mathcal{D}} \approx \prod_i \frac{|\mu_i|}{\mu_i} \tanh\left(\frac{\pi\,|\mu_i|}{2M}\right) \times \qquad (18)$$
$$\times \prod_{\{\text{doublers}\}} \prod_i \frac{|\mu_i|}{\mu_i} \tanh\left(\frac{\pi\,|\mu_i|}{2M_*}\right),$$

where μ_i are eigenvalues of the continuum (chiral) Dirac operator and M_* is some effective cutoff mass of the order $M\Lambda^2 a^2 \sim \sqrt{a}$. We wrote the regularized Dirac operator determinant as a product of determinants of 16 Dirac operators. Each of them goes to ± 1 as a goes to 0 but their product is always $+1$.

This lead to vanishing of the Witten anomaly as it is seen from the lattice. In fact on a lattice there is no reason for Witten anomaly since the topology of the lattice is different from one of the continuum.

Now let us note that the cancelation of the phase factor here is identic to the mechanism proposed by Slavnov in ref.[9] but in this case cancelation arises "naturally" from the lattice. The role of fields introduced in ref.[9] to cancel Witten anomaly is played here by the lattice doubling states.

Discussion

In the present work we proposed a chiral gauge invariant lattice regularization of $SU(2)$ YM model with correct (perturbative) continuum limit. The method includes using a second scale regularization with infinite number of Pauli–Villars fields [4] and insertion of a high order derivative term into fermionic action.

There is no doubling or other lattice artefact on the perturbative level. But despite vanishing perturbatively, the doubler's contribution produces the Witten anomaly cancelation mechanism proposed in [9].

Another lecture we learned from this is that a lattice model which is perturbatively equivalent to some continuum model can, however, be different from it at nonperturbative level. Moreover, since the only way of

[2] Moreover, due to hypercubic invariance it is $\propto 2^4 = 16$.

defining nongaussian path integral, beyond the perturbation theory is the lattice discretization one is tempted to define nonperturbative features of the continuum model as an extrapolation of the lattice one.

Since for a given lattice spacing the momentum function $P_\mu(p)$ is bounded from above one can limit oneself to a finite number of PV fields since the fields with $Mr \gg \frac{1}{a}$ will decouple. In this case decay of the integrand near the edge of the Brillouin zone will be polynomial rather than exponential. The same arguments and conclusions, however, will remain valid for this choice. But for practical calculation it seems to be more convenient to use the Grassmannian even path integral where the gauge invariant Wilson term can be constructed for calculation of inverse (absolute value of) fermionic determinant.

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Appendix

In this appendix we will derive the expression (18) for the determinant. Since the gauge fields U_μ are external ones for sufficiently small lattice spacings a they can be chosen close to the unity. Also let T^A, $(A = 1, ..., 15)$ be generators of the symmetry which relates fermionic doubling states (see e.g. [11])

$$\{T^A\} = \{I, T^\mu, T^\mu T^\nu \ (\nu > \mu), ..., T^1 T^2 T^3 T^4\}, \tag{A1}$$

where

$$T^\mu = i\gamma_\mu \gamma_5 (-)^{\frac{x^\alpha}{a}},$$

commutes with the naive Dirac operator

$$\widehat{D} T^A = T^A \widehat{D}$$

This symmetry is responsible for fermion doubling. Laplace operator $\frac{1}{\Lambda^2}\Delta$ is not invariant under this symmetry[3] but it transforms as follows

$$T_\alpha^{-1} \Delta T_\alpha = \Delta + \frac{1}{\Lambda^2 a^2} \left(U_\alpha(x) \psi_{x+a\widehat{\alpha}} + U_{-\alpha}(x) \psi_{x-a\widehat{\alpha}} \right) \tag{A2}$$

[3] This is also the reason why Wilsonian term kills fermion doubling states.

the second term under our conditions is approximately $\frac{1}{\Lambda^2 a^2}\psi_x$.

Now consider the modified Dirac operator $\widehat{\mathcal{D}} = \left(\widehat{D} + \frac{1}{2\Lambda^2}\{\widehat{D}, \Delta\}\right)$ and consider eigenvalues of $\widehat{\mathcal{D}}$ lying in the subspace of small eigenvalues of Δ, $\lambda(\Delta) \ll 1$. The spectrum of the modified Dirac operator $\widehat{\mathcal{D}}$ here is close to the spectrum of the naive one (and for small lattice spacings also to the spectrum of the continuum Dirac operator). If now to act on such an eigenstate with eigenvalue μ_i by a transformation T the resulting state will have eigenvalue close to $\frac{1}{\Lambda^2 a^2}\mu_i$. This way one finds fifteen such states. For greater values of $\widehat{D}$ this relation becomes less and less exact but greater values are cut off and they do not contribute (except the sign) to the determinant (18). The extra factor $\sim \frac{1}{\Lambda^2 a^2}$ can be included in the cut off mass by its rescaling

$$M \rightarrow M_* = M\Lambda^2 a^2. \tag{A3}$$

Finally let us note that the requirement that $U_\mu \rightarrow 1$ as $a \rightarrow 0$ is important here. This is natural since gauge fields are external. But if one wants to consider dynamical gauge fields here one has to introduce a regularization with a cut off of the order $\sim 1/\sqrt{a}$ also for the gauge fields. This can be easily seen from the perturbation theory if one considers e.g. self-energy Feynman diagram for the fermionic field. Since PV mechanism cannot be applied here one has to have a cutoff for the gauge fields to get rid of doubling contribution of the fermionic propagator.

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